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# DENOMINATORS FOR THE POINCARÉ SERIES OF INVARIANTS OF SMALL MATRICES*

BY

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ABSTRACT

We determine explicit denominators for the Poincaré series of (a) the invariants of $m$ generic $N \times N$ matrices, and (b) the ring generated by $m$ generic $N \times N$ matrices and their traces, for $N \leq 4$. For $N \leq 3$ we prove (and for $N = 4$ we conjecture) that the denominators we obtain are of minimum degree. We also give explicit rational fractions for both series for small values of $m$ and $N$.

## 1. Introduction

In this paper we investigate the Poincaré series for the ring of invariants of an $m$-tuple of $N \times N$ matrices ($N \leq 4$) using results of Van den Bergh [V]. There are a number of equivalent ways to define this series; the one we prefer uses generic matrices. Let $F$ be a field of characteristic 0, and let $X_1, \ldots, X_m$ be $N \times N$ matrices whose entries are independent indeterminates in some commutative $F$-algebra. Let $R$ be the $F$-algebra generated by $X_1, \ldots, X_m$, $\bar{C}$ the algebra generated by traces of elements of $R$, and $\bar{R}$ the algebra generated by $R$ and $\bar{C}$, regarding the members of $\bar{C}$ as scalars. Each of the rings $R$, $\bar{C}$ and $\bar{R}$ has an

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$m$-fold grading by multidegree, and so each has a (symmetric) Poincaré series in the variables $t_1, \ldots, t_m$.

Our interest in this paper will be in the Poincaré series of $\bar{C}$ and $\bar{R}$, both of which are known to be rational functions. We determine explicit denominators for both series for $N \leq 4$ and all $m$, and for $N \leq 3$ prove that these are the denominators of minimum degree. For example, in the case of $3 \times 3$ matrices, the denominator of minimum degree for the Poincaré series of $\bar{C}$ is

$$\prod_i (1-t_i)(1-t_i^2)(1-t_i^3) \prod_{i<j} (1-t_it_j)^2(1-t_i^2t_j)(1-t_it_j^2) \prod_{i<j<k} (1-t_it_jt_k),$$

and for the Poincaré series of $\bar{R}$ it is

$$\prod_i (1-t_i)^2(1-t_i^2) \prod_{i<j} (1-t_it_j)^2(1-t_i^2t_j)(1-t_it_j^2) \prod_{i<j<k} (1-t_it_jt_k).$$

In the case $N = 4$, the denominators we obtain are minimal for $m = 2$, and we conjecture this to be true for all $m$. We also determine explicit rational fractions for both series in the cases $(m, N) = (2,3)$, $(3,3)$, and $(2,4)$. The series for $\bar{C}$ in the cases $(m, N) = (2,3)$ and $(2,4)$ were first obtained by Teranishi in [T1] and [T2], although the latter appears with typographical errors.

## 2. Three formulas

Let $P_N(\bar{C})(t_1, \ldots, t_m)$ and $P_N(\bar{R})(t_1, \ldots, t_m)$ denote the Poincaré series of $m$ generic $N \times N$ matrices with trace, as above. In statements that apply to both series, we will simply write $P_N$. In this section we present three formulas for $P_N(\bar{C})$ and $P_N(\bar{R})$ from the literature. The first two can be found in Section 6 of [F] (an excellent introduction to the subject) and are based on Procesi's work in [P]. The third is due to Van den Bergh and is derived from the second using graph theory.

Let $\lambda$ denote a partition of an integer $n$, $\chi^\lambda$ the corresponding irreducible $S_n$-character, and $s_\lambda(t_1, \ldots, t_m)$ the corresponding Schur function. We use Fr to denote the Frobenius homomorphism, i.e., the linear map from $S_n$-characters to homogeneous symmetric polynomials of degree $n$ in which $\mathrm{Fr}(\chi^\lambda) = s_\lambda(t_1, \ldots, t_m)$. Define an $S_n$-character

$$\theta_N^{(n)} = \sum_{\mu=(\mu_1,\mu_2,\ldots,\mu_N)} \chi^\mu \otimes \chi^\mu,$$

where the sum ranges over partitions of $n$ with at most $N$ parts, and the tensor denotes pointwise product (i.e., the operation corresponding to tensor products of $S_n$-modules).

THEOREM 2.1: *We have*

$$P_N(\bar{C})(t_1,\dots,t_m) = \sum_{n\geq 0} \mathrm{Fr}(\theta_N^{(n)})(t_1,\dots,t_m),$$
$$P_N(\bar{R})(t_1,\dots,t_m) = \sum_{n\geq 0} \mathrm{Fr}(\theta_N^{(n+1)}\downarrow_{S_n})(t_1,\dots,t_m).$$

Both series are also expressible as integrals over the torus

$$\{(z_1,\dots,z_N)\in \mathbf{C}^N : |z_i| = 1\}$$

with respect to Haar measure $d\nu = (2\pi\sqrt{-1})^{-N}(z_1\cdots z_N)^{-1}dz_1\wedge\cdots\wedge dz_N$ as follows.

THEOREM 2.2: *If $|t_1|<1,\dots,|t_m|<1$, then we have*

$$P_N(\bar{C})(t_1,\dots,t_m) = \frac{1}{N!}\int \frac{\prod_{i\neq j}(1-z_iz_j^{-1})}{\prod_{i,j,k}(1-z_iz_j^{-1}t_k)}\,d\nu,$$
$$P_N(\bar{R})(t_1,\dots,t_m) = \frac{1}{N!}\int \frac{\prod_{i\neq j}(1-z_iz_j^{-1})}{\prod_{i,j,k}(1-z_iz_j^{-1}t_k)}\sum_{i,j} z_iz_j^{-1}\,d\nu.$$

The third theorem is the main tool of this paper. It is based on Theorem 2.2 and expresses $P_N$ as the limit of an explicit but complicated sum of rational functions. To describe this sum, define $\Gamma$ to be the directed graph with vertices $\{1,\dots,N\}$ and edges $e(\alpha,\beta,k)$ $(1\leq\alpha\neq\beta\leq N,\ 1\leq k\leq m)$ directed from $\alpha$ to $\beta$. Let $\mathcal{T}$ denote the set of spanning trees of $\Gamma$ such that every vertex has a directed path in the tree to vertex 1.

Choose indeterminates $t(\alpha,\beta,k)$ for each edge $e(\alpha,\beta,k)$ and, for any subset $E$ of the edge set of $\Gamma$, define $w(E)$ to be the product of $t(\alpha,\beta,k)$ over all $e(\alpha,\beta,k)\in E$.

For each tree $T\in\mathcal{T}$, let $T_\alpha$ denote the set of edges in the (unique) path in $T$ from vertex $\alpha$ to vertex 1, and define

$$f_T = \prod_{\alpha\neq\beta}(1-w(T_\beta)/w(T_\alpha)),\quad g_T = \sum_\alpha w(T_\alpha)\cdot\sum_\alpha 1/w(T_\alpha),$$
$$h_T = \prod_{e(\alpha,\beta,k)\notin T}(1-t(\alpha,\beta,k)w(T_\beta)/w(T_\alpha)).$$

Here then is Van den Bergh's result (see Theorem 5.5 of [V]):

THEOREM 2.3: *For $m, N \geq 2$ but not $(m, N) = (2, 2)$, we have*

$$P_N(\bar{C})(t_1, \ldots, t_m) = \frac{1}{N!} \prod_i \frac{1}{(1-t_i)^N} \cdot \lim_{t(\alpha,\beta,k) \to t_k} \sum_{T \in \mathcal{T}} f_T/h_T,$$

$$P_N(\bar{R})(t_1, \ldots, t_m) = \frac{1}{N!} \prod_i \frac{1}{(1-t_i)^N} \cdot \lim_{t(\alpha,\beta,k) \to t_k} \sum_{T \in \mathcal{T}} g_T f_T/h_T.$$

When $m = N = 2$, the formula for $P_N(\bar{C})$ is valid, but the one for $P_N(\bar{R})$ is not.

## 3. Denominators

The use of Theorem 2.3 is complicated by the fact that the individual summands $f_T/h_T$ and $g_T f_T/h_T$ may be singular in the limit $t(\alpha, \beta, k) \to t_k$. However, we shall see that it is possible to partition $\mathcal{T}$ into small equivalence classes so that the sum of the terms indexed by each equivalence class has a nonsingular limit.

We adopt the convention that if $f$ is any rational function of the variables $t(\alpha, \beta, k)$, then $\bar{f}$ shall denote the limit of $f$ as $t(\alpha, \beta, k) \to t_k$ (assuming this limit exists). In particular, if $E$ is a subset of the edge set of $\Gamma$, then $\bar{w}(E)$ is the product of $t_k$ as $e(\alpha, \beta, k)$ ranges over $E$.

We say that a subgraph $S$ of $\Gamma$ is **isotropic** if

(1) There are no directed cycles in $S$.
(2) Vertex 1 is the unique sink of $S$.
(3) For every directed path $E$ in $S$, $\bar{w}(E)$ depends only on the endpoints of $E$, not the particular path.

Note that since $\bar{w}(E)$ is a monomial of degree $|E|$, it follows that in an isotropic graph, all directed paths from a given vertex to the sink have the same length.

It is clear that every tree $T \in \mathcal{T}$ is isotropic.

Given any isotropic subgraph $S$ of $\Gamma$, we define the **isotropic closure** of $S$ to be

$$S^* = \{e(\alpha, \beta, k) \in \Gamma \colon \bar{w}_\alpha = t_k \bar{w}_\beta\},$$

where $\bar{w}_\alpha$ denotes the common value of $\bar{w}(E)$ for all directed paths $E$ in $S$ from $\alpha$ to 1. It is easy to see that $S^*$ is also an isotropic subgraph of $\Gamma$, and that it is maximal with respect to this property. We define $\mathcal{S}$ to be the set of all such maximal isotropic graphs, i.e., the closures of all isotropic subgraphs of $\Gamma$.

We now collect the terms in Theorem 2.3 corresponding to all trees with a given isotropic closure by defining

$$P_S(\bar{C})(t_1,\dots,t_m) = \lim_{t(\alpha,\beta,k)\to t_k} \sum_{T\in\mathcal{T}:T\subseteq S} f_T/h_T,$$
$$P_S(\bar{R})(t_1,\dots,t_m) = \lim_{t(\alpha,\beta,k)\to t_k} \sum_{T\in\mathcal{T}:T\subseteq S} g_T f_T/h_T,$$

for each $S\in\mathcal{S}$. The following result shows that both limits are nonsingular.

LEMMA 3.1: *For all $S\in\mathcal{S}$, both $P_S(\bar{C})$ and $P_S(\bar{R})$ are rational functions of $t_1,\dots,t_m$ with denominator**

$$\prod_{e(\alpha,\beta,k)\notin S} (1-t_k\bar{w}_\beta/\bar{w}_\alpha)^{n_{\alpha\beta}}, \tag{3.1}$$

*where $n_{\alpha\beta}$ denotes the number of pairs of directed paths in $S$ from $\alpha$ to $\gamma$ and $\beta$ to $\gamma$ that intersect only at $\gamma$, for some vertex $\gamma$.*

*Proof:* More generally, we show that the result is true if the numerator of the term indexed by $T$ (i.e., $f_T$ or $g_Tf_T$) is replaced by any Laurent polynomial $p_T$ in the variables $w(T_1),\dots,w(T_N)$. In any such case, it is clear that a denominator for a sum of terms $p_T/h_T$ can be obtained by taking the least common multiple of the terms $h_T$. Each factor appearing in each term $h_T$ is of the form $1-u$, where $u$ is a Laurent monomial in the variables $t(\alpha,\beta,k)$ with exponents $0,\pm1$. All such (distinct) monomials are pairwise co-prime, except for pairs of the form $1-u$, $1-u^{-1}$. Note that there is no term $h_T$ that contains both $1-u$ and $1-u^{-1}$ as factors, since each factor involves a monomial with exactly one variable $t(\alpha,\beta,k)$ corresponding to an edge not in $T$, and this variable occurs with a positive exponent.

We claim that if $1-u\to 0$ in the limit $t(\alpha,\beta,k)\to t_k$, then there is a pairing of trees $T$ that have $1-u$ as a factor of $h_T$ with trees $T'$ that have $1-u^{-1}$ as a factor of $h_{T'}$ so that $1-u$ can be omitted from a common denominator for $p_T/h_T+p_{T'}/h_{T'}$. Indeed, suppose $T\in\mathcal{T}$ is a spanning tree of $S$ such that $h_T$ includes the factor $1-u$. Thus $u=t(\alpha,\beta,k)w(T_\beta)/w(T_\alpha)$ for some (unique) edge $e(\alpha,\beta,k)\notin T$. Since $u\to 1$, it follows that $e(\alpha,\beta,k)\in S$. Let $T'\in\mathcal{T}$ denote the spanning tree of $S$ obtained by deleting from $T$ the unique edge $e(\alpha,\beta',k')$ with tail $\alpha$, and replacing it with the edge $e(\alpha,\beta,k)$. It follows that $h_{T'}$ includes the factor $1-u^{-1}=1-t(\alpha,\beta',k')w(T_{\beta'})/w(T'_\alpha)$, and it is not hard to see that $T\mapsto T'$ is an involution.

* In the sense of ratios of Laurent polynomials.

We have $w(T'_\gamma) = u \cdot w(T_\gamma)$ or $w(T'_\gamma) = w(T_\gamma)$ according to whether the unique path in $T$ (or equivalently, $T'$) from $\gamma$ to 1 passes through the vertex $\alpha$. Therefore

$$\frac{p_T}{h_T} + \frac{p_{T'}}{h_{T'}} = \frac{Q(1)}{1-u} + \frac{Q(u)}{1-u^{-1}} = \frac{Q(1) - uQ(u)}{1-u} \tag{3.2}$$

for some rational function $Q(u)$ that depends on $u$ and the variables $t(\cdot,\cdot,\cdot)$ and is nonsingular at $u = 1$. However $Q(1) - uQ(u) \to 0$ in the limit $u \to 1$, so $1 - u$ can be omitted from a common denominator for $p_T/h_T + p_{T'}/h_{T'}$, which proves the claim.

It follows that a common denominator for $\sum p_T/h_T$ (summed over $T \in \mathcal{T}$ with $T \subseteq S$) consists of the product of all distinct expressions of the form $1 - t(\alpha,\beta,k)w(T_\beta)/w(T_\alpha)$ that tend to a nonzero limit as $t(\alpha,\beta,k) \to t_k$ (i.e., $e(\alpha,\beta,k) \notin S$). Since $T_\alpha$ and $T_\beta$ are paths directed towards the root of a tree, they coincide as soon as they reach a common vertex $\gamma$, and $w(T_\beta)/w(T_\alpha)$ depends only on the parts of $T_\alpha$ and $T_\beta$ from $\alpha$ and $\beta$ to $\gamma$. Hence there is one such expression corresponding to each edge $e(\alpha,\beta,k) \notin S$ and each of the $n_{\alpha\beta}$ pairs of directed paths in $S$ from $\alpha$ and $\beta$ that are disjoint until they reach a common and terminal vertex $\gamma$. Each such factor $1 - t(\alpha,\beta,k)w(T_\beta)/w(T_\alpha)$ converges to $1 - t_k\bar{w}_\beta/\bar{w}_\alpha$, so in the limit we obtain a rational function of $t_1,\ldots,t_m$ with the claimed denominator. ∎

By Theorem 2.3 and the previous lemma, we may obtain a denominator for $P_N$ by taking the least common multiple of all expressions of the form (3.1) as $S$ ranges over $\mathcal{S}$, together with $\prod_i (1-t_i)^N$. The denominators so obtained will be far from minimal in general. However, the following pair of results will allow us to delete some (but not all) of the unnecessary factors.

Lemma 3.2 (cf. the remark prior to Prop. 3.2 in [T2]): *The degree of the pole at $t_i = 1$ in $P_N(t_1,\ldots,t_m)$ is at most $N$.*

*Proof:* It follows from Theorem 2.2 that in the limit $t_i \to 1$,

$$(1-t_i)^N P_N(t_1,\ldots,t_m)$$

is a rational function of the remaining variables. ∎

Thus we may disregard all factors of the form $1 - t_i^{\pm 1}$ that appear in (3.1), provided that we include $(1-t_i)^N$ as a factor of any claimed denominator for $P_N$.

The following result is Proposition 5.1 of [V].

LEMMA 3.3: *$P_N(t_1,\ldots,t_m)$ has a denominator expressible as a product of factors of the form $1-u$, where each $u$ is a true monomial (i.e., not Laurent) of degree $\le N$ in the variables $t_1,\ldots,t_m$.*

Thus we may disregard all factors $1-t_k\bar{w}_\beta/\bar{w}_\alpha$ that appear in (3.1) other than those for which $\bar{w}_\alpha$ divides $t_k\bar{w}_\beta$, or vice-versa.

## 4. Numerators

Let $G_N(\bar{C})$ and $G_N(\bar{R})$ denote denominators for $P_N(\bar{C})$ and $P_N(\bar{R})$ (respectively), and let $F_N(\bar{C})$ and $F_N(\bar{R})$ be the corresponding numerators. By Lemma 3.3, we may assume

$$G_N(t_1,\ldots,t_m)=\prod_{u\in\mathcal{M}}(1-u) \tag{4.1}$$

where $\mathcal{M}$ is a multiset of monomials of degree $\le N$ (depending on the choice of $\bar{C}$ or $\bar{R}$) that is invariant under permutations of the variables $t_1,\ldots,t_m$.

LEMMA 4.1: *For $m,N\ge 2$ but not $(m,N)=(2,2)$, the numerator $F_N(t_1,\ldots,t_m)$ is a polynomial of degree $m(d-N^2)$ that satisfies the functional equation*

$$F_N(t_1^{-1},\ldots,t_m^{-1})=(-1)^{N(m-1)+k-1}(t_1\cdots t_m)^{N^2-d}F_N(t_1,\ldots,t_m),$$

*where $d$ denotes the degree of $G_N(t_1,\ldots,t_m)$ as a polynomial in $t_1$, and $k=|\mathcal{M}|$ denotes the number of factors appearing in (4.1).*

*Proof:* For all $T\in\mathcal{T}$, the substitutions $t(\alpha,\beta,k)\to 1/t(\alpha,\beta,k)$ leave both $f_T$ and $g_T$ invariant, whereas

$$h_T\to(-1)^{N-1}h_T\cdot\prod_{e(\alpha,\beta,k)\in\Gamma}t(\alpha,\beta,k).$$

Thus it follows from Theorem 2.3 that

$$P_N(t_1^{-1},\ldots,t_m^{-1})=(-1)^{N(m-1)-1}(t_1\cdots t_m)^{N^2}P_N(t_1,\ldots,t_m)$$

for both $\bar{C}$ and $\bar{R}$. (In [T1], Teranishi proves this for $\bar{C}$ directly from the integral representation of Theorem 2.2.) Similarly, it is clear from (4.1) that

$$G_N(t_1^{-1},\ldots,t_m^{-1})=(-1)^k u^{-1}G_N(t_1,\ldots,t_m)$$

for some monomial $u$. However $G_N(t_1^{-1}, \ldots, t_m^{-1})$ and $G_N(t_1, \ldots, t_m)$ are both symmetric, so $u$ must be a symmetric monomial, i.e., $u = (t_1 \cdots t_m)^d$. Thus follows the claimed functional equation for $F_N$. Furthermore, since the constant term of $F_N$ is 1, the symmetry implied by the functional equation shows that $\pm(t_1 \cdots t_m)^e$ (where $e = d - N^2$) must be the dominant term of $F_N$, which proves the claimed expression for the degree. ∎

In order to determine explicit formulas for the Poincaré series $P_N$, our strategy is to first use Theorem 2.3 and Lemma 3.1 to obtain a suitable denominator $G_N$. We then use Theorem 2.1 to compute $P_N$ through terms of degree $n = \lfloor m(d - N^2)/2 \rfloor$. This requires a calculation involving the irreducible characters of the symmetric groups up to degree $n$ for $\bar{C}$ and degree $n+1$ for $\bar{R}$. We then compute $F_N = G_N \cdot P_N$ through terms of degree $n$ and deduce the remaining terms from the functional equation of Lemma 4.1.

## 5. 2 × 2 Matrices

PROPOSITION 5.1: *The Poincaré series for* $2 \times 2$ *matrices have denominators*

$$G_2(\bar{C})(t_1, \ldots, t_m) = \prod_i (1 - t_i) \prod_{i \le j} (1 - t_i t_j),$$
$$G_2(\bar{R})(t_1, \ldots, t_m) = \prod_i (1 - t_i)^2 \prod_{i < j} (1 - t_i t_j).$$

*Proof:* Recall that for $2 \times 2$ matrices, Theorem 2.3 is valid only for $m \ge 3$. However there is no loss of generality in assuming $m \ge 3$, since $P_N(t_1, \ldots, t_{m-1}) = P_N(t_1, \ldots, t_{m-1}, 0)$, and the same is true of the claimed denominators.

The trees $T \in \mathcal{T}$ each consist of a single edge $e(2,1,i)$ for some $i$ $(1 \le i \le m)$. In particular, all such trees are maximally isotropic, so $\mathcal{S} = \mathcal{T}$. Ignoring factors of $1 - t_i$ (see Lemma 3.2), the tree $T = \{e(2,1,i)\}$ indexes a term that contributes a rational function with denominator

$$\bar{h}_T = \prod_{k \ne i} (1 - t_k/t_i) \prod_k (1 - t_i t_k),$$

by Lemma 3.1. We may disregard $1 - t_k/t_i$ by Lemma 3.3, and the remaining factors divide $G_2(\bar{C})$, so $G_2(\bar{C})$ is a denominator for $P_2(\bar{C})$. In the case of $\bar{R}$, the factor $1 - t_i^2$ does not divide $G_2(\bar{R})$, however the corresponding numerator is $\bar{f}_T \cdot \bar{g}_T = (1 - t_i^2)(1 - 1/t_i^2)$. ∎

Applying Lemma 4.1, we obtain the following.

COROLLARY 5.2: *For $m \geq 3$, we have $P_2 = F_2/G_2$, where $F_2(t_1, \ldots, t_m)$ is a symmetric polynomial of degree $m(m-2)$ (for $\bar{C}$), or $m(m-3)$ (for $\bar{R}$). Moreover,*

$$F_2(t_1, \ldots, t_m) = (-1)^{\binom{m}{2}-1}(t_1 \cdots t_m)^e F_2(t_1^{-1}, \ldots, t_m^{-1}),$$

*where $e = m-2$ (for $\bar{C}$) or $m-3$ (for $\bar{R}$).*

COROLLARY 5.3: *We have*

$$P_2(\bar{C})(t_1, t_2, t_3) = \frac{1 + t_1 t_2 t_3}{\prod_i (1 - t_i) \prod_{i \leq j} (1 - t_i t_j)},$$

$$P_2(\bar{R})(t_1, t_2, t_3) = \frac{1}{\prod_i (1 - t_i)^2 \prod_{i < j} (1 - t_i t_j)}.$$

*Proof:* By Corollary 5.2, the numerators for $m = 3$ have degree 3 (for $\bar{C}$) and 0 (for $\bar{R}$). The latter immediately yields $F_2(\bar{R}) = 1$. In the former case, it is immediate from the definition that the linear term of $P_2(\bar{C})$ is $\sum t_i$. On the other hand, the linear term of $G_2(\bar{C})$ is $-\sum t_i$, so the numerator has no linear term and the remaining terms of $F_2(\bar{C})$ can be inferred from the functional equation of Corollary 5.2. ∎

The Poincaré series $P_2(\bar{C})$ and $P_2(\bar{R})$ were first determined explicitly for $m = 2$ by Formanek, Halpin and Li [FHL], and then $m = 4$ by Formanek in [F]. Also in [F], Formanek notes that the Schur function expansion of $P_N(t_1, \ldots, t_m)$ only involves partitions with $N^2$ rows, so in the case $N = 2$, $m = N^2 = 4$ variables are sufficient to determine the Poincaré series for all $m$.

In fact, we claim that $N(N-1)$ variables are sufficient. By Theorem 2.2,

$$P_N(t_1, \ldots, t_m) = \prod_i \frac{1}{(1 - t_i)^N} \cdot Q_N(t_1, \ldots, t_m),$$

where $Q_N(t_1, \ldots, t_m)$ denotes the constant term in the expansion of

$$p(z_1, \ldots, z_N) \prod_{i \neq j, k} (1 - z_i z_j^{-1} t_k)^{-1}$$

in terms of "rational" Schur functions $(z_1 \cdots z_N)^{-r} s_\lambda(z_1, \ldots, z_N)$, and $p = 1$ (for $\bar{C}$) or $p = \sum_{i,j} z_i z_j^{-1}$ (for $\bar{R}$). It follows immediately from the Cauchy identity (e.g., [M, §I.4]) that the Schur function expansion of $Q_N(t_1, \ldots, t_m)$ only involves partitions with at most $N(N-1)$ rows. In the case $N = 2$, this means that $P_2(t_1, t_2)$ contains sufficient information to determine $P_2(t_1, \ldots, t_m)$ for all $m$.

COROLLARY 5.4: *We have*

$$P_2(\bar{C})(t_1,\dots,t_m) = \prod_i \frac{1}{(1-t_i)^2} \sum_{a\geq b\geq 0} (-1)^{a+b} s_{(a,b)}(t_1,\dots,t_m),$$

$$P_2(\bar{R})(t_1,\dots,t_m) = \prod_i \frac{1}{(1-t_i)^2} \sum_{a\geq 0} s_{(a,a)}(t_1,\dots,t_m),$$

*Proof:* By Corollary 5.3, we have

$$Q_2(\bar{C})(t_1,t_2) = \frac{1}{(1+t_1)(1+t_2)(1-t_1t_2)} = \sum_{a\geq b\geq 0} (-1)^{a+b} s_{(a,b)}(t_1,t_2),$$

$$Q_2(\bar{R})(t_1,t_2) = \frac{1}{1-t_1t_2} = \sum_{a\geq 0} s_{(a,a)}(t_1,t_2),$$

using the fact that $s_{(a,b)}(t_1,t_2) = t_1^a t_2^b + t_1^{a-1}t_2^{b+1} + \cdots + t_1^b t_2^a$. ∎

A determinantal formula for $P_2(\bar{R})(t_1,\dots,t_m)$ has been given by Le Bruyn [L].

*Remark 5.5:* For all $m \geq 1$, the denominator of minimum degree for $P_2(\bar{C})(t_1,\dots,t_m)$ is $G_2(\bar{C})(t_1,\dots,t_m)$. Indeed, if $G_2$ failed to be minimal, then by symmetry, the numerator $F_2$ would have to be divisible by one of $1-t_1$, $1+t_1$, or $1-t_1t_2$. Furthermore, since $P_N(t_1,\dots,t_k,0,\dots,0) = P_N(t_1,\dots,t_k)$, any divisor of $F_N(t_1,\dots,t_m)$ depending only on $t_1,\dots,t_k$ would also be a divisor of $F_N(t_1,\dots,t_k)$. However $F_2(\bar{C})(t_1,t_2) = 1$ (e.g., by Corollary 5.3), so the claim follows. Similarly, $F_2(\bar{R})(t_1,t_2) = 1$, so $G_2(\bar{R})(t_1,\dots,t_m)$ is the denominator of minimum degree for $P_2(\bar{R})(t_1,\dots,t_m)$.

## 6. 3 × 3 Matrices

Let $[u] := 1-u$ and $[u]_n := (1-u)(1-u^2)\cdots(1-u^n)$.

PROPOSITION 6.1: *The Poincaré series for* $3\times 3$ *matrices have denominators*

$$G_3(\bar{C})(t_1,\dots,t_m) = \prod_i [t_i]_3 \prod_{i<j} [t_it_j]^2[t_i^2t_j][t_it_j^2] \prod_{i<j<k} [t_it_jt_k],$$

$$G_3(\bar{R})(t_1,\dots,t_m) = \prod_i [t_i][t_i]_2 \prod_{i<j} [t_it_j]^2[t_i^2t_j][t_it_j^2] \prod_{i<j<k} [t_it_jt_k].$$

*Proof:* Following the technique outlined in Section 3, we first examine the trees in $\mathcal{T}$. The members can be grouped into three types:

$$A(i,j) = \{e(2,1,i), e(3,1,j)\},$$
$$B(i,j) = \{e(3,2,i), e(2,1,j)\}, \text{ and}$$
$$C(i,j) = \{e(2,3,i), e(3,1,j)\},$$

where $1 \leq i, j \leq m$. Each of these trees is maximally isotropic, so $\mathcal{S} = \mathcal{T}$.

Since $G_3(\bar{C})$ and $G_3(\bar{R})$ both have zeros of multiplicity 3 at $t_i = 1$, it follows that for each $S \in \mathcal{T}$, we may disregard all factors in (3.1) of the form $1 - t_i^{\pm 1}$ (Lemma 3.2). Thus we need only to examine factors of the form $1 - u^{\pm 1}$ where $u$ is a monomial of degree $\geq 2$ (Lemma 3.3).

In the case $S = A(i, j)$, the product of these factors is

$$\prod_k (1 - t_i t_k)(1 - t_j t_k).$$

This divides both $G_3(\bar{C})$ and $G_3(\bar{R})$ unless $i = j$. However in that case, we have $\bar{w}(S_2) = \bar{w}(S_3) = t_i$, so $\bar{f}_S = 0$. That is, the numerator for the term indexed by $S$ vanishes.

In the case $S = B(i, j)$ or $S = C(i, j)$, the analogous product is

$$\prod_k (1 - t_i t_k)(1 - t_j t_k)(1 - t_i t_j t_k). \tag{6.1}$$

Again, this divides both $G_3(\bar{C})$ and $G_3(\bar{R})$ unless $i = j$. In that case, the product includes both $(1 - t_i^2)^2$, which divides neither claimed denominator, and $1 - t_i^3$, which does not divide $G_3(\bar{R})$. However, in the case $i = j$ we have $\bar{w}(S_3)/\bar{w}(S_1) = t_i^2$ or $\bar{w}(S_2)/\bar{w}(S_1) = t_i^2$, so the corresponding numerators $\bar{f}_S$ and $\bar{f}_S \cdot \bar{g}_S$ are divisible by a factor of $1 - t_i^2$ that can be canceled from (6.1). Furthermore, in the case of $\bar{R}$, we may cancel

$$\bar{g}_S = (1 + t_i + t_i^2)(1 + 1/t_i + 1/t_i^2),$$

against the factor $1 - t_i^3$ appearing in (6.1), leaving a factor $1 - t_i$ whose multiplicity is controlled by Lemma 3.2. ∎

Using Lemma 4.1, we may deduce the degrees of the corresponding numerators.

Corollary 6.2: *For $m \geq 2$, we have $P_3 = F_3/G_3$, where $F_3(t_1, \ldots, t_m)$ is a symmetric polynomial of degree $em$, and $e = (m^2 + 7m - 14)/2$ (for $\bar{C}$), or $e = (m^2 + 7m - 18)/2$ (for $\bar{R}$). Moreover,*

$$F_3(t_1, \ldots, t_m) = (-1)^{\binom{m}{3}} (t_1 \cdots t_m)^e F_3(t_1^{-1}, \ldots, t_m^{-1}).$$

Note that $F_3(\bar{R})$ has degree 0 when $m = 2$, so we immediately obtain

COROLLARY 6.3: *We have*

$$P_3(\bar{R})(t_1,t_2) = \frac{1}{[t_1]^2[t_1^2][t_2]^2[t_2^2][t_1t_2]^2[t_1^2t_2][t_1t_2^2]}.$$

Similarly, $F_3(\bar{C})$ has degree 4 when $m=2$, and hence can be computed using $S_n$ characters with $n \le 2$. For this it is convenient to substitute $t_i \to t_i q$ so that the variable $q$ records total degree. From the definition of $\theta_3^{(n)}$ in Section 2, it is clear that $\theta_3^{(1)}$ is the trivial character of $S_1$ and $\theta_3^{(2)}$ is twice the trivial character of $S_2$, so Theorem 2.1 implies

$$\begin{aligned} P_3(\bar{C})(t_1q,t_2q) &= 1 + s_1(t_1,t_2)q + 2s_2(t_1,t_2)q^2 + O(q^3) \\ &= 1 + (t_1+t_2)q + (2t_1^2 + 2t_1t_2 + 2t_2^2)q^2 + O(q^3). \end{aligned}$$

From the definition of $G_3(\bar{C})$, we have

$$G_3(\bar{C})(t_1q,t_2q) = 1 - (t_1+t_2)q - (t_1^2 + t_1t_2 + t_2^2)q^2 + O(q^3),$$

and hence

$$F_3(\bar{C})(t_1q,t_2q) = G_3(\bar{C})(t_1q,t_2q)\cdot P_2(\bar{C})(t_1q,t_2q) = 1 - t_1t_2q^2 + O(q^3).$$

Combining this with the functional equation of Corollary 6.2 yields

COROLLARY 6.4 (Teranishi [T1]): *We have*

$$P_3(\bar{C})(t_1,t_2) = \frac{1 - t_1t_2 + t_1^2t_2^2}{[t_1]_3[t_2]_3[t_1t_2]^2[t_1^2t_2][t_1t_2^2]}.$$

In the case $m=3$, the Poincaré series have numerators of degree 24 (for $\bar{C}$) and 18 (for $\bar{R}$), so we need the terms of degree 12 and 9 in order to make full use of the functional equation. This is too large to be done by hand, but the computation can be done easily with *SF*,* the second author's Maple package for symmetric functions [S]. The results are as follows.

PROPOSITION 6.5: *We have*

$$\begin{aligned} F_3(\bar{R})(t_1,t_2,t_3) = (1+e_3)(1 &+ e_3 + e_1e_3 - e_2e_3 - e_1e_2e_3 + e_3^2 - e_2^2e_3 + e_1e_3^2 \\ &+ e_1^2e_3^2 - e_2e_3^2 + e_1e_2e_3^2 - e_3^3 + e_1e_3^3 - e_2e_3^3 - e_3^4 - e_3^5), \\ F_3(\bar{C})(t_1,t_2,t_3) = 1 &- e_2 + e_3 + e_1e_3 + e_2^2 + e_1^2e_3 - e_2e_3 - 2e_1e_2e_3 + e_3^2 + e_2^2e_3 \\ &- e_1^2e_2e_3 + 2e_1^2e_3^2 + e_1^3e_3^2 + e_2^2e_3^2 - e_1^2e_2e_3^2 - e_1e_3^3 - 2e_1e_2^2e_3^2 \\ &+ 2e_2e_3^3 - e_2^3e_3^2 + e_1^3e_3^3 + 2e_1^2e_2e_3^3 - 2e_1e_3^4 - e_1^2e_3^4 + e_1e_2^2e_3^3 \\ &+ e_2e_3^4 - e_2^3e_3^3 - 2e_2^2e_3^4 - e_1^2e_3^5 + e_1e_2^2e_3^4 + 2e_1e_2e_3^5 - e_3^6 - e_2^2e_3^5 \\ &+ e_1e_3^6 - e_2e_3^6 - e_1^2e_3^6 - e_3^7 + e_1e_3^7 - e_3^8, \end{aligned}$$

*where $e_1, e_2, e_3$ denote the elementary symmetric functions of $t_1, t_2, t_3$.*

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* Freely available at `http://www.math.lsa.umich.edu/~jrs/maple.html`

*Remark 6.6:* By reasoning similar to Remark 5.5, it follows that for all $m \geq 1$, the denominator of minimum degree for $P_3(\bar{C})(t_1, \ldots, t_m)$ is $G_3(\bar{C})(t_1, \ldots, t_m)$. Indeed, since $F_3(\bar{C})(t_1, t_2)$ and $G_3(\bar{C})(t_1, t_2)$ are relatively prime, this could fail only if $F_3(\bar{C})(t_1, t_2, t_3)$ were divisible by $1 - t_1t_2t_3 = 1 - e_3$. However, one can use Proposition 6.5 to check that $F_3(\bar{C})(t_1, t_2, t_3) = 2(e_1^3 - e_2^3) \bmod 1 - e_3$. Similarly, we have $F_3(\bar{R})(t_1, t_2) = 1$ and $F_3(\bar{R})(t_1, t_2, t_3) = 2(e_1 - e_2)(3 + e_1 + e_2) \bmod 1 - e_3$. Therefore $G_3(\bar{R})(t_1, \ldots, t_m)$ is the denominator of minimum degree for $P_3(\bar{R})(t_1, \ldots, t_m)$.

## 7. 4 × 4 Matrices

THEOREM 7.1: *The Poincaré series for* $4 \times 4$ *matrices have denominators*

$$G_4(\bar{C})(t_1, \ldots, t_m) = \prod_i [t_i]_4 \prod_{i<j} [t_it_j]^2[t_i^2t_j^2][t_i^2t_j]^2[t_it_j^2]^2[t_i^3t_j][t_it_j^3]$$
$$\prod_{i<j<k} [t_it_jt_k]^2[t_i^2t_jt_k][t_it_j^2t_k][t_it_jt_k^2] \prod_{i<j<k<l} [t_it_jt_kt_l],$$
$$G_4(\bar{R})(t_1, \ldots, t_m) = \prod_i [t_i][t_i]_3 \prod_{i<j} [t_it_j]^2[t_i^2t_j^2][t_i^2t_j]^2[t_it_j^2]^2[t_i^3t_j][t_it_j^3]$$
$$\prod_{i<j<k} [t_it_jt_k]^2[t_i^2t_jt_k][t_it_j^2t_k][t_it_jt_k^2] \prod_{i<j<k<l} [t_it_jt_kt_l].$$

*Proof:* We will first prove that

$$\widehat{G}_4(\bar{C}) := G_4(\bar{C}) \cdot \prod_i (1 + t_i) \prod_{i<j} (1 - t_it_j)^3,$$
$$\widehat{G}_4(\bar{R}) := G_4(\bar{R}) \cdot \prod_i (1 + t_i)^2 \prod_{i<j} (1 - t_it_j)^3$$

are denominators for $P_4(\bar{C})$ and $P_4(\bar{R})$. A more delicate argument will then be used to eliminate the "extra" factors.

Consider representatives of the four isomorphism classes of rooted trees in $\mathcal{T}$, say

$$A(i_1, i_2, i_3) = \{e(2, 1, i_1), e(3, 1, i_2), e(4, 1, i_3)\},$$
$$B(i_1, i_2, i_3) = \{e(2, 1, i_1), e(3, 2, i_2), e(4, 2, i_3)\},$$
$$C(i_1, i_2, i_3) = \{e(2, 1, i_1), e(3, 1, i_2), e(4, 2, i_3)\},$$
$$D(i_1, i_2, i_3) = \{e(2, 1, i_1), e(3, 2, i_2), e(4, 3, i_3)\},$$

where $1 \leq i_1, i_2, i_3 \leq m$. The remaining members of $\mathcal{T}$ are obtained from these by permuting the vertices 2, 3 and 4. The only members of this list that are not maximally isotropic are the trees of the form $C(i,i,j)$ and $C(i,j,j)$; their isotropic closures are

$$C^*(i,i,j) = \{e(2,1,i), e(3,1,i), e(4,2,j), e(4,3,j)\},$$
$$C^*(i,j,j) = \{e(2,1,i), e(3,1,j), e(4,2,j), e(4,3,i)\}.$$

In the case $S = A(i_1, i_2, i_3)$, the product of all factors in (3.1) of the form $1 - u^{\pm 1}$ with $u$ a monomial of degree $\geq 2$ is

$$\prod_k (1 - t_{i_1} t_k)(1 - t_{i_2} t_k)(1 - t_{i_3} t_k).$$

If $i_1, i_2, i_3$ are distinct, this clearly divides both $G_4(\bar{C})$ and $G_4(\bar{R})$. Otherwise, if (say) $i_1 = i_2 = i$, then $\bar{w}(S_2) = \bar{w}(S_3) = t_i$, so $\bar{f}_S = 0$, i.e., the corresponding numerators vanish in both the $\bar{C}$ and $\bar{R}$ cases.

In the case $S = B(i_1, i_2, i_3)$, the relevant product is

$$\prod_k (1 - t_{i_1} t_k)(1 - t_{i_2} t_k)(1 - t_{i_3} t_k)(1 - t_{i_1} t_{i_2} t_k)(1 - t_{i_1} t_{i_3} t_k), \tag{7.1}$$

which divides $G_4(\bar{C})$ unless $i_1 = i_2 = i_3$, and divides $G_4(\bar{R})$ unless $i_1, i_2, i_3$ are not distinct. However if $i_2 = i_3$, then $\bar{w}(S_3) = \bar{w}(S_4)$ and $\bar{f}_S = 0$, so again both numerators vanish. If $i_1 = i_2 = i$, then (7.1) includes a factor $(1 - t_i^2)^2$ that fails to divide $G_4(\bar{R})$. However in that case, $\bar{w}(S_3)/\bar{w}(S_1) = t_i^2$, so the numerator includes a factor $1 - t_i^2$ that can be canceled against it.

In the case $S = C(i_1, i_2, i_3)$ with $i_2 \notin \{i_1, i_3\}$ (so that $S$ is maximally isotropic), the relevant product is

$$(1 - t_{i_1} t_{i_3}) \prod_k (1 - t_{i_1} t_k)(1 - t_{i_2} t_k)(1 - t_{i_3} t_k)(1 - t_{i_1} t_{i_3} t_k). \tag{7.2}$$

The factor $1 - t_{i_1} t_{i_3}$ corresponds to the choice of $e(3,4,i_2)$ in (3.1). Indeed, each edge of the form $e(3,4,k)$ contributes the factor $1 - u$, where $u = t_k t_{i_1} t_{i_3} / t_{i_2}$. However $u^{\pm 1}$ is not a monomial unless $k = i_2$. It is easy to check that (7.2) divides both $G_4(\bar{C})$ and $G_4(\bar{R})$ unless $i_1 = i_3 = i$ for some $i$. In that case, $(1 - t_i^2)^3$ divides (7.2), whereas $G_4(\bar{C})$ is only divisible by $(1 - t_i^2)^2$ and $G_4(\bar{R})$ by $1 - t_i^2$. However $\bar{w}(S_4)/\bar{w}(S_1) = t_i^2$, so $\bar{f}_S$ includes $(1 - t_i^2)(1 - 1/t_i^2)$ as a factor that can be canceled against (7.2).

In the case $S = C^*(i,i,j)$, note that there are two trees $T \in \mathcal{T}$ that are subgraphs of $S$, and hence $P_S$ is (a limit of) a sum of two rational functions of the form $f_T/h_T$ (for $\bar{C}$) or $f_Tg_T/h_T$ (for $\bar{R}$). However the numerators corresponding to both trees have a common factor $1 - t(2,1,i)/t(3,1,i)$ that vanishes in the limit $t(\alpha,\beta,k) \to t_k$, so $S$ contributes nothing to either Poincaré series.

In the case $S = C^*(i,j,j)$, we may assume $i \neq j$; otherwise we are in the previous case. Since $n_{1,4} = 2$, it follows that the product of all factors in (3.1) of the form $1 - u^{\pm 1}$ with $u$ a monomial of degree $\geq 2$ is

$$\prod_k (1 - t_it_k)^3(1 - t_jt_k)^3(1 - t_it_jt_k)^2.$$

This divides both $\widehat{G}_4(\bar{C})$ and $\widehat{G}_4(\bar{R})$, but not $G_4(\bar{C})$ and $G_4(\bar{R})$. The failures are caused by the presence of the factors $(1 - t_it_j)^6$ and $(1 - t_i^2)^3(1 - t_j^2)^3$.

In the case $S = D(i_1, i_2, i_3)$, the relevant product is

$$(7.3)\quad a\cdot\prod_k (1-t_{i_1}t_k)(1-t_{i_2}t_k)(1-t_{i_3}t_k)(1-t_{i_1}t_{i_2}t_k)(1-t_{i_2}t_{i_3}t_k)(1-t_{i_1}t_{i_2}t_{i_3}t_k),$$

where $a$ denotes the product of all *distinct* expressions of the form

$$1 - t_{i_1}t_{i_2}, \quad 1 - t_{i_2}t_{i_3}, \quad 1 - t_{i_1}t_{i_3},$$

corresponding to the factors in (3.1) indexed by the edges $e(4,1,k)$ ($k = i_3, i_2, i_1$). If $i_1, i_2, i_3$ are distinct, then it is not hard to check that (7.3) divides $G_4(\bar{C})$ and $G_4(\bar{R})$. If $i_1 = i_2 = i_3 = i$ for some $i$, then (7.3) simplifies to

$$(1 - t_i^2)\prod_k (1 - t_it_k)^3(1 - t_i^2t_k)^2(1 - t_i^3t_k),$$

which fails to divide both $G_4(\bar{C})$ and $G_4(\bar{R})$. However $\bar{w}(S_\alpha) = t_i^{\alpha-1}$ ($1 \leq \alpha \leq 4$), so

$$\begin{aligned} \bar{f}_S &= (1 - t_i)^3(1 - 1/t_i)^3(1 - t_i^2)^2(1 - 1/t_i^2)^2(1 - t_i^3)(1 - 1/t_i^3), \\ \bar{g}_S &= (1 + t_i)(1 + 1/t_i)(1 + t_i^2)(1 + 1/t_i^2). \end{aligned}$$

When these are canceled against (7.3), the results divide $G_4(\bar{C})$ and $G_4(\bar{R})$.

Continuing the hypothesis that $S = D(i_1, i_2, i_3)$, consider the possibility that $i_1 = i_2 = i$ and $i_3 = j$, or $i_1 = j$ and $i_2 = i_3 = i$, for some $i \neq j$. In this case, (7.3) simplifies to

$$(7.4)\quad (1 - t_i^2)(1 - t_it_j)\prod_k (1 - t_it_k)^2(1 - t_jt_k)(1 - t_i^2t_k)(1 - t_it_jt_k)(1 - t_i^2t_jt_k).$$

This is divisible by $(1-t_i^2)^3$ and $(1-t_it_j)^5$, so it fails to divide both $G_4(\bar{C})$ and $G_4(\bar{R})$. However $\bar{f}_S$ includes the factors $(1-t_i^2)(1-1/t_i^2)$ and $(1-t_it_j)(1-1/t_it_j)$. When these are canceled against (7.4), the results do divide $G_4(\bar{C})$ and $G_4(\bar{R})$. On the other hand, if $i_1 = i_3 = i$ and $i_2 = j$, then (7.4) should be modified by replacing $1-t_i^2t_k$ with a second copy of $1-t_it_jt_k$. Again, the factors $(1-t_i^2)^3$ and $(1-t_it_j)^5$ are the only obstructions to divisibility, and again the corresponding numerators include factors of the form $(1-t_it_j)(1-1/t_it_j)$ (in fact, the square of such a factor appears). However in the case of $\bar{C}$, the numerators have no factors of the form $1-t_i^2$ or $1+t_i$. Thus the remaining part of (7.4) divides $G_4(\bar{C})\prod_k(1+t_k)$, but not necessarily $G_4(\bar{C})$. On the other hand,

$$\bar{g}_S = (1+t_i)(1+1/t_i)(1+t_it_j)(1+1/t_it_j),$$

so the remaining part of (7.4) does divide $G_4(\bar{R})$.

The preceding argument shows that $\widehat{G}_4(\bar{C})$ and $\widehat{G}_4(\bar{R})$ are common denominators for each of the rational functions that appear in the expansions of $P_4(\bar{C})$ and $P_4(\bar{R})$. Moreover, the only terms that $G_4(\bar{C})$ and $G_4(\bar{R})$ fail to "denominate" are those indexed by the graphs isomorphic to $C^*(i,j,j)$ (in both cases) or $D(i,j,i)$ (in the case of $\bar{C}$), for some $i \neq j$.

Henceforth, let us fix $i \neq j$, $S = C^*(i,j,j)$, and $T = D(i,j,i)$. To finish, it suffices to prove the following.

(a) $P_S(\bar{R})$ has a pole at $t_i = -1$ of multiplicity at most one.

(b) $P_S(\bar{R})$ and $P_S(\bar{C})$ have denominators that are not divisible by $(1-t_it_j)^4$.

(c) $P_S(\bar{C}) + P_T(\bar{C})$ has a pole at $t_i = -1$ of multiplicity at most two.

To prove (a), let $T' = C(i,j,j)$ and $T''$ denote the two spanning trees of $S$ in $\mathcal{T}$. Recall from the proof of Lemma 3.1 (see (3.2)) that there is a rational function of the variables $u = t(4,3,i)t(3,1,j)/t(4,2,j)t(2,1,i)$ and $t(\alpha,\beta,k)$, say $Q(u)$, such that

$$\frac{g_{T'}f_{T'}}{h_{T'}} = \frac{Q(1)}{1-u}, \quad \text{and} \quad \frac{g_{T''}f_{T''}}{h_{T''}} = \frac{Q(u)}{1-u^{-1}} = \frac{-uQ(u)}{1-u}.$$

Furthermore, $Q(u)$ is nonsingular in the limit $u \to 1$. It follows that

$$P_S(\bar{R}) = \lim_{\substack{t(\alpha,\beta,k)\to t_k \\ u\to 1}} \frac{Q(1)-uQ(u)}{1-u} = \bar{Q}(1) + \bar{Q}'(1),$$

treating $u$ as an independent variable. It is easy to check that $h_{T'}/(1-u)$ has a zero at $t_i = -1$ of multiplicity two in the limit $t(\alpha,\beta,k) \to t_k$. Similarly, the numerator includes

$$\bar{g}_{T'} = (1+t_i)(1+t_j)\cdot(1+1/t_i)(1+1/t_j),$$

so $\bar{Q}(1)$ is nonsingular in the limit $t_i \to -1$. Furthermore, by logarithmic differentiation we may write $\bar{Q}'(1)/\bar{Q}(1)$ as a linear combination of terms of the form $F'(1)/F(1)$, where $F(u)$ ranges over the factors appearing in the numerator and denominator of $\bar{Q}(u)$. However, each such factor $F(u)$ has the property that $F(1)$ has at most a simple zero at $t_i = -1$, so $\bar{Q}'(1)$ must have at most a simple pole at $t_i = -1$.

Similarly, to prove (b) for $\bar{R}$, one can check that in $Q(u)$ there are two factors in the numerator and four factors in the denominator that are divisible by $1-t_it_j$ in the limit $u \to 1$, $t(\alpha,\beta,k) \to t_k$. In $\bar{Q}(u)$, these factors are $(1-ut_it_j)(1-1/ut_it_j)$ in the numerator and $(1-t_it_j)^2(1-ut_it_j)^2$ in the denominator. Hence $\bar{Q}(1)$ and $\bar{Q}'(1)$ both have denominators in which $1-t_it_j$ occurs with multiplicity at most 2. Essentially the same argument applies in the case of $\bar{C}$ as well.

To prove (c), it suffices to show that

$$\lim_{t_i\to -1}(1+t_i)^3 P_T(\bar{C}) = -\lim_{t_i\to -1}(1+t_i)^3 P_S(\bar{C}), \tag{7.5}$$

since we have already shown that $P_S(\bar{C})$ and $P_T(\bar{C})$ both have poles of order at most 3 at $t_i = -1$. From the definition, we obtain

$$P_T = \frac{[t_i]^2[1/t_i]^2[t_j][1/t_j][t_it_j]^2[1/t_it_j]^2[t_i^2t_j][1/t_i^2t_j]}{\prod_k [t_it_k]^2[t_jt_k][t_it_jt_k]^2[t_i^2t_jt_k][t_k/t_it_j]^2[t_k/t_i^2t_j]\prod_{k\neq i}[t_k/t_i]^2\prod_{k\neq j}[t_k/t_j]}.$$

Taking into account the vanishing factors $[t_i^2]^2[1/t_i^2]$, it follows that

$$\lim_{t_i\to -1}(1+t_i)^3 P_T = -2\cdot\prod_{k\neq i}\frac{1}{[-t_k]^4[t_jt_k]^2[-t_jt_k]^2[-t_k/t_j]^2}\prod_{k\neq i,j}\frac{1}{[t_k/t_j]^2}. \tag{7.6}$$

To compute the analogous limit for $P_S$, note that by reasoning similar to (a), we have

$$P_S(\bar{C}) = \bar{Z}(1) + \bar{Z}'(1),$$

where $Z(u)$ is the rational function of $u$ and $t(\alpha,\beta,k)$ such that

$$\frac{f_{T'}}{h_{T'}} = \frac{Z(1)}{1-u}, \quad\text{and}\quad \frac{f_{T''}}{h_{T''}} = \frac{Z(u)}{1-u^{-1}},$$

and $u, T', T''$ are as defined above. Since $\bar{Z}(1)$ has a pole of order 2 at $t_i = -1$, it contributes nothing to the coefficient of $(1+t_i)^{-3}$ in the Laurent expansion of $P_S$. Furthermore, by logarithmic differentiation it follows that $\bar{Z}'(1)$ is $\bar{Z}(1)$ times a linear combination of expressions of the form $F'(1)/F(1)$, where $F(u)$ ranges over the factors appearing in the numerator and denominator of $\bar{Z}(u)$.

There is exactly one factor, namely, $F(u) = 1 - ut_i^2$ such that $F'(1) \neq 0$ and $F(1)$ vanishes at $t_i = -1$, and it appears in the denominator. Hence this is the unique factor such that $F'(1)/F(1)$ is singular at $t_i = -1$, it has a residue of $-1/2$, and (as a factor of the denominator) it occurs with coefficient $-1$. Thus

$$\lim_{t_i \to -1} (1+t_i)^3 P_S(\bar{C}) = (1/2) \lim_{t_i \to -1} (1+t_i)^2 \bar{Z}(1).$$

Since $Z(1) = (1-u) f_{T'}/h_{T'}$, we find

$$\bar{Z}(1) = \frac{[t_i]^2[1/t_i]^2[t_j]^2[1/t_j]^2[t_it_j][1/t_it_j][t_i/t_j][t_j/t_i]}{\prod_k [t_it_k]^2[t_jt_k]^2[t_it_jt_k][t_k/t_it_j][t_kt_j/t_i][t_kt_i/t_j] \prod_{k\neq i}[t_k/t_i]^2 \prod_{k \neq j}[t_k/t_j]^2},$$

and therefore

$$\lim_{t_i \to -1} (1+t_i)^2 \bar{Z}(1) = 4 \cdot \prod_{k \neq i} \frac{1}{[-t_k]^4[t_jt_k]^2[-t_jt_k]^2[-t_k/t_j]^2} \prod_{k \neq i,j} \frac{1}{[t_k/t_j]^2}.$$

Comparing this with (7.6), we obtain (7.5). ∎

Again via Lemma 4.1, we obtain the following.

Corollary 7.2: *For $m \geq 2$, we have $P_4 = F_4/G_4$, where $F_4(t_1, \ldots, t_m)$ is a symmetric polynomial of degree $em$, and $e = \binom{m-1}{3} + 6\binom{m-1}{2} + 14(m-1) - 6$ (for $\bar{C}$), or $e = \binom{m-1}{3} + 6\binom{m-1}{2} + 14(m-1) - 9$ (for $\bar{R}$). Moreover,*

$$F_4(t_1, \ldots, t_m) = (-1)^{k-1}(t_1 \cdots t_m)^e F_4(t_1^{-1}, \ldots, t_m^{-1}),$$

*where $k = \binom{m}{2} + \binom{m}{3} + \binom{m}{4}$.*

In particular, $F_4(\bar{C})(t_1, t_2)$ and $F_4(\bar{R})(t_1,t_2)$ have degrees 16 and 10 and can be determined by a Maple computation of their terms of degree $\leq 8$ and $\leq 5$ (respectively) via Theorem 2.1. The results are as follows.

Proposition 7.3: *We have*

$$P_4(\bar{C})(t_1,t_2) = \frac{(1 - t_1t_2 + t_1^2t_2^2)(1 - e_1t_1t_2 + e_1t_1^2t_2^2 + e_1^2t_1^2t_2^2 + e_1t_1^3t_2^3 - e_1t_1^4t_2^4 + t_1^6t_2^6)}{[t_1]_4[t_2]_4[t_1t_2]^2[t_1t_2^2]^2[t_1^2t_2]^2[t_1t_2^3][t_1^3t_2][t_1^2t_2^2]},$$

$$P_4(\bar{R})(t_1,t_2) = \frac{1 + t_1^2t_2^2 + t_1^2t_2^3 + t_1^3t_2^2 + t_1^3t_2^3 + t_1^5t_2^5}{[t_1][t_1]_3[t_2][t_2]_3[t_1t_2]^2[t_1t_2^2]^2[t_1^2t_2]^2[t_1t_2^3][t_1^3t_2][t_1^2t_2^2]},$$

*where $e_1 = t_1 + t_2$.*

We remark that Teranishi calculated $P_4(\bar{C})(t_1,t_2)$ in [T2] (Theorem 4.1), although with a denominator that is a multiple of the one we use here. There are also typographical errors in the numerator (for example, the formula as printed is not symmetric).

We conclude with two conjectures.

CONJECTURE 7.4: *$G_4(\bar{C})$ and $G_4(\bar{R})$ are the denominators of minimum degree for $P_4(\bar{C})$ and $P_4(\bar{R})$.*

Since $F_4(t_1,t_2)$ and $G_4(t_1,t_2)$ are relatively prime (for both $\bar{C}$ and $\bar{R}$), it follows that this conjecture could fail only if $1-t_1t_2t_3$ or $1-t_1^2t_2t_3$ divides $F_4(t_1,t_2,t_3)$, or $1-t_1t_2t_3t_4$ divides $F_4(t_1,\ldots,t_4)$.

For all $N \geq 1$, let $D_N(t_1,\ldots,t_m)$ denote the denominator of minimum degree for the Poincaré series of either $\bar{C}$ or $\bar{R}$.

CONJECTURE 7.5: *We have*

(a) $D_N(t_1,\ldots,t_m) = (1-u_1)\cdots(1-u_k)$, *where each $u_i$ is a monomial of degree $\leq N$.*

(b) $D_N(t_1,\ldots,t_m,0) = D_N(t_1,\ldots,t_m)$.

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